# Derivation of the Expected Covariance Matrix and Means Vector of $Z = bXY + e$ using Products of Variables

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Abstract

This short paper is a supplement to the article "Products of Variables in Structural Equation Models".

The Products of Variables article derives the variance of a product of two variables using the methods of moments. However, it neglects to derive the full expected covariance matrix of the model $Z = b_1XY + e$. This supplement to the article derives the full expected covariance matrix of $Z = bXY + e$ using the method of moments for products of variables. We first derive the expected covariance assuming all variables are centered in order to show why this model is underdetermined and thus has an infinite number of equally likely maximum likelihood solutions. We next derive the expected means and covariances of the model for the case where the multiplicands have nonzero means and show that this model does have a single maximum likelihood solution. Some of this supplement is redundant with material found in the main article, but this material is included here so that the supplement can be read as a stand alone short paper.

## Expectations by Method of Moments for Products of Centered Variables

Suppose we are given an SEM as a path diagram in which all nodes with incoming edges are additionally labeled as sum or product of their incoming edges. For the time being, we will assume that the graph is acyclic (recursive). It may be that the results can be extended to cases with cycles as well, provided that for all node values, a infinite series calculated along each cycle converges.

We are interested in all moments of the vector of all observed variables. The following describes how those can be computed analytically assuming a fixed set of parameters. Wall and Amemiya Wall and Amemiya (2001, 2003) proposed to represent variance sources of an

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SEM as independent standard-normally distributed sources. We have extended and generalized this idea by transforming SEM models represented in Reticular Action Model (RAM, McArdle & McDonald, 1984) format into an equivalent models that have only monomial, independent standard-normally distributed variables. Then, all moments become expectations of polynomials, which in turn are sums of monomials. In this way, the computation of the moments is reduced to computing the expectation of a monomial of independently standard-normally distributed variables–a problem that has a known computational solution.

At the highest level, the algorithm proceeds in three steps,

1. All variables in the SEM are represented by a linear combination of some independently normally distributed variables $w_1, ..., w_n$ with known variances such that the covariance matrix of all variables is the symmetrical matrix $\mathbf{S}$ from the RAM matrix formulation.

2. Progressing top-down in the asymmetrical graph of the path diagram, polynomial representations of all variables in the $w_1, ..., w_n$ are computed.

3. Polynomial representations of all requested moments are computed and evaluated into numbers.

Suppose that we have an SEM represented in standard RAM notation such that the, the model-expected covariance matrix of the observed variables, $\mathbf{C}_{XX}$ is calculated as

$$\mathbf{C}_{XX} = \mathbf{F}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}((\mathbf{I} - \mathbf{A})^{-1})^T\mathbf{F}^T \tag{1}$$

where for all variables, both latent and manifest, $\mathbf{A}$ is the matrix of regression coefficients, $\mathbf{S}$ is the matrix of variances and covariances, and $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{F}$ filters out the latent variables so that $\mathbf{C}_{XX}$ contains only the model-expected covariance matrix of the observed variables.

In order to transform the model into a model with only independent variables, we will operate on $\mathbf{S}$, the matrix of model variances and covariances both latent and manifest. We first compute the Eigenvalue decomposition of $\mathbf{S}$,

$$\mathbf{S} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \tag{2}$$

Let $W = w_1, ..., w_n$ be independently normally distributed variables with zero mean and variances given by the diagonal entries of $\mathbf{D}$, the Eigenvalues of $\mathbf{S}$, where $n$ is the number of total variables in the SEM. Then $\mathbf{Q}W$ is an $n$-dimensional random variable with zero mean and covariance

$$\mathbb{V}(\mathbf{Q}W) = \mathbb{E}(\mathbf{Q}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{Q}^T) = \mathbf{S} \tag{3}$$

So each variable can be expressed by a linear combination of the $W$ variables with the corresponding row of $Q$ as weights, plus a constant term that gives the mean of that variable.

Using the product and sum nodes, every variable in the model (both observed and latent) $X_i$ can now be represented as a polynomial $f_i$ in $w_1, ..., w_n$. If there are $m$ variables in total, the $k = (k_1, ..., k_m)$-th moment of the joint distribution of the vector $X = (x_1, ..., x_m)$ is

$$M_k(X) = \mathbb{E}(\prod_{i=1}^{m} f_i^{k_i}) \tag{4}$$

where the expectation is taken with respect to the roots $w_i$. In particular, $M_k(X)$ is the expected value of a polynomial in $w_i$, where the coefficients are combination of the regression weights in the SEM. Let this polynomial be $g = \prod_{i=1}^{m} f_i^{k_i}$. This polynomial is a sum of monomials in $w_i$. Since the expectation is linear, we can separate the computation of the expectation to the single monomials, and thus reduce our problem to computing the expectation of a monomial of independently normally distributed variables, i.e., an expectation of the form

$$\mathbb{E}\left(w_1^{e_1} \cdots w_n^{e_n}\right) \tag{5}$$

which is given by the product of the expectations of the single variables with their exponents,

$$\mathbb{E}\left(w_1^{e_1} \cdots w_n^{e_n}\right) = \prod_{i=1}^{n} \mathbb{E}(w_i^{e_i}) \tag{6}$$

This last equation is not completely trivial, and so for completeness sake, we give a quick proof:

**Theorem 1.**

$$\mathbb{E}\left(W_1^{e_1} \cdots W_n^{e_n}\right) = \prod_{i=1}^{n} \mathbb{E}(W_i^{e_i}) \tag{7}$$

*Proof.*

$$\begin{aligned}
\mathbb{E}\left(W_1^{e_1} \cdots W_n^{e_n}\right) &= \int_{-\infty}^{\infty} w_1^{e_1} \cdots w_n^{e_n} pdf(w_1) \cdots pdf(w_n) dw_1 \dots dw_n \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} w_1^{e_1} pdf(w_1) dw \right) w_2^{e_2} \cdots w_n^{e_n} pdf(w_2) \cdots pdf(w_n) dw_2 \dots dw_n \\
&= \mathbb{E}(W_1^{e_1}) \int_{-\infty}^{\infty} w_2^{e_2} \cdots w_n^{e_n} pdf(w_2) \cdots pdf(w_n) dw_2 \dots dw_n \\
&\vdots \\
&= \prod_{i=1}^{n} \mathbb{E}(W_i^{e_i})
\end{aligned}$$

□

Thus, we are left with computing the higher-order moments of independent standard-normally distributed variables with zero mean and unit variance,

$$\mathbb{E}(W^e) \,, \tag{8}$$

where $e$ is an integer. These moments are known (e.g., Papoulis & Pillai, 2002) to be

$$\mathbb{E}(W^e) = \left\{ \begin{array}{ll} 0 & \text{if } e \text{ is odd} \\ \mathbb{V}(W)^{e/2} \prod_{i=1}^{\frac{e-2}{2}} (2i+1) & \text{if } e \text{ is even} \end{array} \right\} \tag{9}$$

**Variance of $Z$**

We first transform the SEM model for $Z = bXY + e$ such that all sources of variance and covariances are independent, normally distributed variables with mean zero and unit variance. The first step is to remove all covariances between variables by replacing them with unit variance sources. Thus, we add the latent variable $w_2$ and replace the covariance path with the value $\mathbb{C}(XY)$ in Figure 1-a with regression paths from $w_2$ to $X$ and $Y$. Thus, the covariance between $X$ and $Y$ can be calculated as $((\mathbb{C}(XY))^{\frac{1}{2}} \cdot 1 \cdot (\mathbb{C}(XY))^{\frac{1}{2}})) = \mathbb{C}(XY)$. However, now the variances for $X$ and $Y$ become residual variances that must be reduced by the total effect of $w_2$ which is $(\mathbb{C}(XY)^{\frac{1}{2}}) \cdot 1 \cdot (\mathbb{C}(XY)^{\frac{1}{2}}) = \mathbb{C}(XY)$. So, the residual variance of $X$ is $V_X - \mathbb{C}(XY)$ and the residual variance of $Y$ is $V_Y - \mathbb{C}(XY)$.

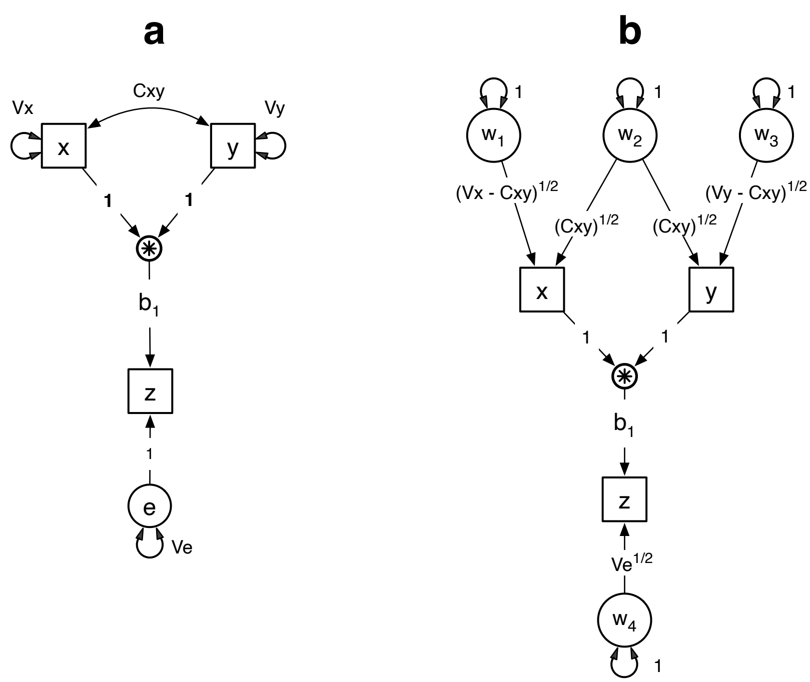


*Figure 1*. Path diagrams of a product of two variables. The asterisk surrounded by a circle represents the product of the variables connected to it by incoming arrows. (a) Mean-centered variables with a single product term. (b) Equivalent path diagram with variances sources isolated to be independent normally distributed variables with mean zero and variance one.

We can now replace the variance terms in Figure 1-a with the independent normally distributed unit variance variables $w_1$, $w_3$, and $w_4$ and regression paths to $X$, $Y$, and $e$ respectively. The regression weights for these variables become the square root of the residual variances for $X$, $Y$, and $e$ as shown in Figure 1-b. The $i$th variable in the path diagram in Figure 1-b can now be represented as a polynomial $f_i$ of what we will refer to as *root nodes*, i.e., the independent, normally distributed variables $w_1, \ldots, w_n$ with zero mean and unit variance. This transformation of an SEM model into polynomials of root nodes can be applied to any SEM that can be represented as a RAM model, including models that have n-ary operators such as introduced here.

In the following, we will derive the expected covariance matrix of the diagram above. We first transform the model into the equivalent monomial form model shown in Figure 1-b. To shorten our derivation, we refer to Equations 10, 11, 12, and 13 to define four variables A, B, C, and D.

$$A = w_1(\mathbb{V}(X) - \mathbb{C}(XY))^{\frac{1}{2}} \tag{10}$$
$$B = w_2(\mathbb{C}(XY))^{\frac{1}{2}} \tag{11}$$
$$C = w_3(\mathbb{V}(Y) - \mathbb{C}(XY))^{\frac{1}{2}} \tag{12}$$
$$D = w_4(\mathbb{V}(e))^{\frac{1}{2}}. \tag{13}$$

The expected value of a monomial with an odd power is zero and so $\mathbb{E}(A) = \mathbb{E}(B) = \mathbb{E}(C) = \mathbb{E}(D) = \mathbb{E}(A^3) = \mathbb{E}(B^3) = \mathbb{E}(C^3) = \mathbb{E}(D^3) = 0$. The squares of these four monomials are non-zero and so it is also convenient to define their expected values as

$$V_A = \mathbb{E}(A^2) = \mathbb{V}(X) - \mathbb{C}(XY) \tag{14}$$
$$V_B = \mathbb{E}(B^2) = \mathbb{C}(XY) \tag{15}$$
$$V_C = \mathbb{E}(C^2) = \mathbb{V}(Y) - \mathbb{C}(XY) \tag{16}$$
$$V_D = \mathbb{E}(D^2) = \mathbb{V}(e). \tag{17}$$

Finally, we will need the expected value of the fourth power of B,

$$F_B = \mathbb{E}(B^4) = 3\mathbb{C}(XY)^2 \tag{18}$$

Now we can refer to the three variables in the model using an abbreviated version of their monomial forms such that

$$\begin{aligned} X &= A + B \\ Y &= B + C \\ Z &= b_1XY + D = b_1(A + B)(B + C) + D. \end{aligned} \tag{19}$$

All expectations of monomials with at least one variable to the power of an odd number will result in zero and so we will ignore them.

The second moments can be formed as follows. Since the variances and covariances can be computed by the second moment minus the product of the expectations and the expected value of a variable with mean zero is zero, thus the variances and covariances of centered variables reduce to the second moments. We begin with the moments not including

$Z$:

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}(A^2 + B^2) \\
&= V_A + V_B \\
&= \mathbb{V}(X) - \mathbb{C}(XY) + \mathbb{C}(XY) \\
&= \mathbb{V}(X) && (20) \\
\mathbb{E}(Y^2) &= \mathbb{E}(B^2 + C^2) \\
&= V_B + V_C \\
&= \mathbb{C}(XY) + \mathbb{V}(Y) - \mathbb{C}(XY) \\
&= \mathbb{V}(Y) && (21) \\
\mathbb{E}(XY) &= \mathbb{E}(B^2) \\
&= V_B \\
&= \mathbb{C}(XY) && (22)
\end{aligned}$$

The second moments of variables including $Z$ are

$$\begin{aligned}
\mathbb{E}(XZ) &= \mathbb{E}((A+B)(b_1(A+B)(B+C)+D)) \\
&= \mathbb{E}((A+B)(b_1(AB+B^2+BC)+D)) \\
&= \mathbb{E}(b_1(A^2B+AB^2+ABC)+AD)+(b_1(B^2A+B^3+B^2C)+BD))
\end{aligned}$$

Since the expectation of a constant is that constant we have

$$\mathbb{E}(XZ) = b_1\mathbb{E}(A^2B+AB^2+ABC+AD+B^2A+B^3+B^2C+BD)$$

And now, substituting in the expected values from above we have

$$\begin{aligned}
\mathbb{E}(XZ) &= b_1\mathbb{E}(V_A(0)+(0)V_B+(0)(0)(0)+(0)(0)+V_B(0)+(0)+V_B(0)+(0)(0)) \\
\mathbb{E}(XZ) &= \mathbb{C}(XZ) = 0 && (23)
\end{aligned}$$

By the same logic,

$$\mathbb{E}(YZ) = \mathbb{C}(YZ) = 0 \qquad (24)$$

Deriving the second moments of $Z$ is somewhat more complicated.

$$\begin{aligned}
\mathbb{E}(Z^2) &= \mathbb{E}((b_1(A+B)(B+C)+D)^2) \\
&= \mathbb{E}((b_1(AB+BB+AC+BC)+D)^2) \\
&= \mathbb{E}((b_1AB+b_1BB+b_1AC+b_1BC+D)^2) \\
&= \mathbb{E}((b_1^2A^2B^2+b_1^2B^3A+b_1^2A^2CB+b_1^2B^2CA+b_1DAB+b_1^2B^3+b_1^2B^4+ \\
&\quad bb_1^2B^2AC+b_1^2BB^3C+b_1B^2D+b_1^2A^2CB+b_1^2ACBB^2+b_1^2A^2C^2+b_1^2AC^2B+ \\
&\quad bb_1ACD+b_1^2B^2CA+b_1^2B^3C+b_1^2BC^2A+b_1^2B^2C^2+b_1BCD+ \\
&\quad bDb_1AB+Db_1B^2+Db_1AC+Db_1BC+D^2)) && (25)
\end{aligned}$$

And now, substituting in the expected values from above we have

$$\begin{aligned}\mathbb{E}(Z^2) &= b_1^2 V_A V_B + b_1^2(0)(0) + b_1^2 V_A(0)(0) + b_1^2 V_B(0)(0) + b_1(0)(0)(0) + b_1^2(0) + \\ &\quad b_1^2 F_B + b_1^2 V_B(0)(0) + b_1^2(0)(0) + b_1 V_B(0) + b_1^2 V_A(0)(0) + b_1^2(0)(0)V_B + b_1^2 V_A V_C + \\ &\quad b_1^2(0)V_C(0) + b_1(0)(0)(0) + b_1^2 V_B(0)(0) + b_1^2(0)(0) + b_1^2(0)V_C(0) + b_1^2 V_B V_C + \\ &\quad b_1(0)(0)(0) + b_1(0)(0)(0) + (0)b_1 V_B + (0)b_1(0)(0) + (0)b_1(0)(0) + D^2 \end{aligned} \tag{26}$$

Removing the zeros and collecting terms we find

$$\mathbb{E}(Z^2) = b_1^2(V_A V_B + F_B + V_A V_C + V_B V_C) + D^2 \tag{27}$$

Now we substitute in the values from Equations 34, 35, 36, and 37 to find

$$\begin{aligned}\mathbb{E}(Z^2) &= b_1^2((\mathbb{V}(X) - \mathbb{C}(XY))(\mathbb{C}(XY)) + 3\mathbb{C}(XY)^2 + \\ &\quad (\mathbb{V}(X) - \mathbb{C}(XY))(\mathbb{V}(Y) - \mathbb{C}(XY)) + (\mathbb{C}(XY))(\mathbb{V}(Y) - \mathbb{C}(XY))) + \mathbb{V}(e) \\ &= b_1^2(\mathbb{V}(X)\mathbb{C}(XY) - \mathbb{C}(XY)^2 + 3\mathbb{C}(XY)^2 + \mathbb{V}(X)\mathbb{V}(Y) - \mathbb{C}(XY)\mathbb{V}(Y) - \\ &\quad \mathbb{V}(X)\mathbb{C}(XY) + \mathbb{C}(XY)^2 + \mathbb{C}(XY)\mathbb{V}(Y) - \mathbb{C}(XY)^2) + \mathbb{V}(e) \\ &= b_1^2(\mathbb{V}(X)\mathbb{V}(Y) + \mathbb{C}(XY)^2) + \mathbb{V}(e) \end{aligned} \tag{28}$$

which is the result given by Goodman (1960) and Bohrnstedt and Marwell (1978), who calculated the variance of a product of two zero mean normally distributed variables as $\mathbb{V}(Z) = \mathbb{V}(X)\mathbb{V}(Y) + \mathbb{C}(XY)^2$.

In the derivation of the variance of the outcome variable $Z$ shown in Equation 27, note that almost all of the terms in the expansion end up being zero. In fact, the lower moments of variables that originate from products of normals are often zero, because the $k$th moment of a standard-normally distributed variable is zero if $k$ is odd.

## Expected covariance matrix of $Z = b_1XY + e$

We now have all of the elements in $\mathbb{E}(\Sigma)$, the expected covariance matrix of the model $Z = b_1XY + e$

$$\mathbb{E}(\Sigma) = \begin{bmatrix} V_X & \mathbb{C}(XY) & 0 \\ \mathbb{C}(XY) & V_Y & 0 \\ 0 & 0 & b_1^2(V_X V_Y + \mathbb{C}(XY)^2) + V_e \end{bmatrix} \tag{29}$$

In linear regression, the regression coefficients appear both in the variance of the outcome variable as well as in the covariances between the predictors and outcomes. This determines the regression coefficients. However, in this product of variables model, the regression coefficient only appears in one cell of the expected covariance matrix: the expected variance of the product. Given that the variance of the residual is also only determined by its summation in the variance of the product, the model is underdetermined: the value of the variance of $e$ and the value of $b_1$ can trade off with one another. In addition, since $b_1$ only appears as $b_1^2$ in the expected covariance, the sign of $b_1$ is also underdetermined.

## Expectations for Products of Variables with Nonzero Means

When the means of the predictor variables are included in the model, these means show up in several places in the expected covariance matrix as well as the expected means vector. These means provide sufficient constraints to determine the solution. Consider the product of variables model with means shown in Figure 2-a.

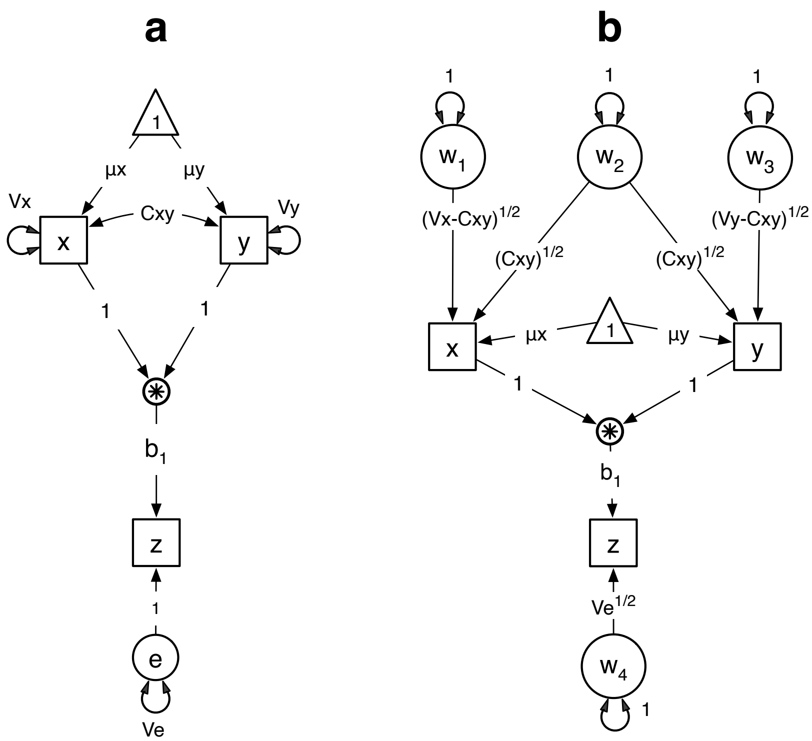


*Figure 2*. Path diagrams of a product of two variables with estimated means for the multiplicands. The asterisk surrounded by a circle represents the product of the variables connected to it by incoming arrows. (a) $X$ and $Y$ with estimated means and a single product term. (b) Equivalent path diagram with variance sources isolated to be independent normally distributed variables with mean zero and variance one.

In the following, we will derive the expected covariance matrix and means vector of the diagram above. We first transform the model into the equivalent monomial form model shown in Figure 2-b. Once again, to shorten our derivation, we refer to Equations 30,31,32, and 33 to define four variables A, B, C, and D.

$$A = w_1(\mathbb{V}(X) - \mathbb{C}(XY))^{\frac{1}{2}} \quad (30)$$

$$B = w_2(\mathbb{C}(XY))^{\frac{1}{2}} \quad (31)$$

$$C = w_3(\mathbb{V}(Y) - \mathbb{C}(XY))^{\frac{1}{2}} \quad (32)$$

$$D = w_4(\mathbb{V}(e))^{\frac{1}{2}}. \quad (33)$$

We also will need to define the expected values of the squares of these four monomials,

$$\begin{aligned} V_A &= \mathbb{E}(A^2) = \mathbb{V}(X) - \mathbb{C}(XY) & (34) \\ V_B &= \mathbb{E}(B^2) = \mathbb{C}(XY) & (35) \\ V_C &= \mathbb{E}(C^2) = \mathbb{V}(Y) - \mathbb{C}(XY) & (36) \\ V_D &= \mathbb{E}(D^2) = \mathbb{V}(e). & (37) \end{aligned}$$

Finally, we will need the expected value of the fourth power of B,

$$\begin{aligned} F_B &= \mathbb{E}(B^4) = 3\mathbb{C}(XY)^2 & (38) \\ & & (39) \end{aligned}$$

Now we can refer to the three variables in the model using an abbreviated version of their monomial forms such that

$$\begin{aligned} X &= A + B + \mu_X \\ Y &= B + C + \mu_Y \\ Z &= b_1 XY + D = b_1(A + B + \mu_X)(B + C + \mu_Y) + D. & (40) \end{aligned}$$

All expectations of monomials with at least one variable to the power of an odd number will result in zero and so we will ignore them. This leaves us with following expectations for the means,

$$\begin{aligned} \mathbb{E}(X) &= \mu_X & (41) \\ \mathbb{E}(Y) &= \mu_Y & (42) \\ \mathbb{E}(Z) &= b_1\mathbb{E}(B^2) + b_1\mu_X\mu_Y = b_1(V_B + \mu_X\mu_Y). & (43) \end{aligned}$$

The second moments can be formed analogously. We begin with the moments not including $Z$:

$$\begin{aligned} \mathbb{E}(X^2) &= \mathbb{E}(A^2 + B^2 + \mu_X^2) \\ &= V_A + V_B + \mu_X^2 \\ &= V_X - \mathbb{C}(XY) + \mathbb{C}(XY) + \mu_X^2 \\ &= V_X + \mu_X^2 & (44) \\ \mathbb{E}(Y^2) &= \mathbb{E}(B^2 + C^2 + \mu_Y^2) \\ &= V_B + V_C + \mu_Y^2 \\ &= \mathbb{C}(XY) + V_Y - \mathbb{C}(XY) + \mu_Y^2 \\ &= V_Y + \mu_Y^2 & (45) \\ \mathbb{E}(XY) &= \mathbb{E}(B^2 + \mu_X\mu_Y) \\ &= V_B + \mu_X\mu_Y \\ &= \mathbb{C}(XY) + \mu_X\mu_Y & (46) \end{aligned}$$

The second moments of variables including $Z$ are

$$\begin{aligned}
\mathbb{E}(XZ) &= \mathbb{E}((A+B+\mu_X)(b_1(A+B+\mu_X)(B+C+\mu_Y)+D)) \\
&= b_1\mathbb{E}(A^2\mu_Y + B^2\mu_Y + 2B^2\mu_X + \mu_X^2\mu_Y) \\
&= b_1(V_A\mu_Y + V_B(\mu_Y + 2\mu_X) + \mu_X^2\mu_Y) \qquad (47) \\
\mathbb{E}(YZ) &= b_1(V_C\mu_X + V_B(\mu_X + 2\mu_Y) + \mu_Y^2\mu_X) \qquad (48)
\end{aligned}$$

Again, deriving the second moments of $Z$ is more complicated.

$$\begin{aligned}
\mathbb{E}(Z^2) &= \mathbb{E}((b_1(A+B+\mu_X)(B+C+\mu_Y)+D)^2) \\
&= b_1^2\mathbb{E}((A^2+B^2+\mu_X^2+2AB+2A\mu_Y+2B\mu_Y) \\
&\quad \cdot(B^2+C^2+\mu_Y^2+2BC+2B\mu_X+2C\mu_X)) + D^2 \\
&= b_1^2\mathbb{E}(A^2B^2 + A^2C^2 + A^2\mu_Y^2 + 2A^2BC + 2A^2B\mu_X + 2A^2C\mu_X + \\
&\quad B^4 + B^2C^2 + B^2\mu_Y^2 + 2B^3C + 2B^3\mu_X + 2B^2C\mu_X + \\
&\quad B^2\mu_X^2 + C^2\mu_X^2 + \mu_X^2\mu_Y^2 + 2BC\mu_X^2 + 2B\mu_X\mu_X^2 + 2C\mu_X\mu_Y^2 + \\
&\quad 2AB^3 + 2ABC^2 + 2AB\mu_Y^2 + 4AB^2C + 4AB^2\mu_X + 4ABC\mu_X + \\
&\quad 2AB^2\mu_Y + 2AC^2\mu_Y + 2A\mu_Y^3 + 4ABC\mu_Y + 4AB\mu_X\mu_Y + 4AC\mu_X\mu_Y + \\
&\quad 2B^3\mu_Y + 2BC^2\mu_Y + 2B\mu_Y^3 + 4B^2C\mu_Y + 4B^2\mu_X\mu_Y + 4BC\mu_X\mu_Y)) + D^2
\end{aligned} \qquad (49)$$

Since anything multiplied by odd powers of A, B, C, or D will be zero, the previous expression can be simplified to

$$\begin{aligned}
\mathbb{E}(Z^2) &= b_1^2\mathbb{E}(A^2B^2 + A^2C^2 + A^2\mu_Y^2 + B^4 + B^2C^2 + B^2\mu_Y^2 + \\
&\quad B^2\mu_X^2 + C^2\mu_X^2 + \mu_X^2\mu_Y^2 + 4B^2\mu_X\mu_Y)) + D^2 \qquad (50)
\end{aligned}$$

Substituting the squared and fourth power monomials we have

$$\begin{aligned}
\mathbb{E}(Z^2) &= b_1^2(V_AV_B + V_AV_C + V_BV_C + F_B + V_A\mu_Y^2 + V_C\mu_X^2 \\
&\quad + V_B(\mu_X^2 + \mu_Y^2 + 4\mu_X\mu_Y) + \mu_X^2\mu_Y^2) + V_D \qquad (51)
\end{aligned}$$

Substituting the expected values of the second and fourth moments we find

$$\begin{aligned}
\mathbb{E}(Z^2) &= b_1^2((\mathbb{V}(X) - \mathbb{C}(XY))\mathbb{C}(XY) + \\
&\quad (\mathbb{V}(X) - \mathbb{C}(XY))(\mathbb{V}(Y) - \mathbb{C}(XY)) + \\
&\quad \mathbb{C}(XY)(\mathbb{V}(Y) - \mathbb{C}(XY)) + 3\mathbb{C}(XY)^2 + \\
&\quad (\mathbb{V}(X) - \mathbb{C}(XY))\mu_Y^2 + (\mathbb{V}(Y) - \mathbb{C}(XY))\mu_X^2 \\
&\quad + \mathbb{C}(XY)(\mu_X^2 + \mu_Y^2 + 4\mu_X\mu_Y) + \mu_X^2\mu_Y^2) + \mathbb{V}(e) \qquad (52) \\
&= b_1^2(\mathbb{V}(X)\mathbb{C}(XY) - \mathbb{C}(XY)^2 + \\
&\quad \mathbb{V}(X)\mathbb{V}(Y) - \mathbb{C}(XY)\mathbb{V}(Y) - \mathbb{V}(X)\mathbb{C}(XY) + \mathbb{C}(XY)^2 + \\
&\quad \mathbb{C}(XY)\mathbb{V}(Y) - \mathbb{C}(XY)^2 + 3\mathbb{C}(XY)^2 + \\
&\quad \mathbb{V}(X)\mu_Y^2 - \mathbb{C}(XY)\mu_Y^2 + \mathbb{V}(Y)\mu_X^2 - \mathbb{C}(XY)\mu_X^2 + \\
&\quad \mathbb{C}(XY)\mu_X^2 + \mathbb{C}(XY)\mu_Y^2 + 2\mathbb{C}(XY)\mu_X\mu_Y + \mu_X^2\mu_Y^2) + \mathbb{V}(e) \qquad (53) \\
&= b_1^2(\mathbb{V}(X)\mu_Y^2 + \mathbb{V}(Y)\mu_X^2 + 2\mathbb{C}(XY)\mu_X\mu_Y + \mu_X^2\mu_Y^2 + \\
&\quad \mathbb{V}(X)\mathbb{V}(Y) + 2\mathbb{C}(XY)^2) + \mathbb{V}(e) \qquad (54)
\end{aligned}$$

The variances and covariances can be computed by the second moment minus the product of the expectations. Again, consider the variances and covariances not including $Z$ first:

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= V_A + V_B + \mu_X^2 - \mu_X^2 \\
&= V_A + V_B \\
&= \mathbb{V}(X) - \mathbb{C}(XY) + \mathbb{C}(XY) \\
&= \mathbb{V}(X) && (55) \\
\mathbb{V}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\
&= V_B + V_C + \mu_Y^2 - \mu_Y^2 \\
&= \mathbb{V}(Y) - \mathbb{C}(XY) + \mathbb{C}(XY) \\
&= \mathbb{V}(Y) && (56) \\
\mathbb{C}(XY) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\
&= V_B + \mu_X\mu_Y - \mu_X\mu_Y \\
&= V_B \\
&= \mathbb{C}(XY) && (57)
\end{aligned}$$

The covariances that include $Z$ are

$$\begin{aligned}
\mathbb{C}(XZ) &= \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) && (58) \\
&= b_1(V_A\mu_Y + V_B(\mu_Y + 2\mu_X) + \mu_X^2\mu_Y) - \mu_X b_1(V_B + \mu_X\mu_Y) \\
&= b_1(V_A\mu_Y + V_B(\mu_Y + 2\mu_X) + \mu_X^2\mu_Y - \mu_X V_B - \mu_X^2\mu_Y) \\
&= b_1(V_A\mu_Y + V_B(\mu_X + \mu_Y)) \\
&= b_1(\mathbb{V}(X)\mu_Y - \mathbb{C}(XY)\mu_Y + \mathbb{C}(XY)\mu_X + \mathbb{C}(XY)\mu_Y) \\
&= b_1(\mathbb{V}(X)\mu_Y + \mathbb{C}(XY)\mu_X) && (59) \\
\mathbb{C}(YZ) &= \mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z) \\
&= b_1(V_C\mu_X + V_B(\mu_X + 2\mu_Y) + \mu_Y^2\mu_X) - \mu_Y b_1(V_B + \mu_Y\mu_X) \\
&= b_1(V_C\mu_X + V_B(\mu_X + 2\mu_Y) + \mu_Y^2\mu_X - \mu_Y V_B - \mu_Y^2\mu_X) \\
&= b_1(V_C\mu_X + V_B(\mu_X + \mu_Y)) \\
&= b_1(\mathbb{V}(Y)\mu_X - \mathbb{C}(XY)\mu_X + \mathbb{C}(XY)\mu_X + \mathbb{C}(XY)\mu_Y) \\
&= b_1(\mathbb{V}(Y)\mu_X + \mathbb{C}(XY)\mu_Y) && (60)
\end{aligned}$$

Finally, the variance of $Z$ is

$$
\begin{aligned}
\mathbb{V}(Z) &= \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 && (61)\\
&= b_1^2(V_A V_B + V_A V_C + V_B V_C + F_B + V_A \mu_Y^2 + V_C \mu_X^2 \\
&\quad + V_B(\mu_X^2 + \mu_Y^2 + 4\mu_X \mu_Y) + \mu_X^2 \mu_Y^2) + V_D \\
&\quad - b_1^2(V_B^2 + 2V_B \mu_X \mu_Y + \mu_X^2 \mu_Y^2) && (62)\\
&= b_1^2(V_A V_B + V_A V_C + V_B V_C + F_B + V_A \mu_Y^2 + \\
&\quad V_C \mu_X^2 - V_B^2 - 2V_B \mu_X \mu_Y - \mu_X^2 \mu_Y^2) + V_D && (63)\\
&= b_1^2((\mathbb{V}(X) - \mathbb{C}(XY))\mathbb{C}(XY) + (\mathbb{V}(X) - \mathbb{C}(XY))(\mathbb{V}(Y) - \mathbb{C}(XY)) + \\
&\quad \mathbb{C}(XY)(\mathbb{V}(Y) - \mathbb{C}(XY)) + 3\mathbb{C}(XY)^2 + (\mathbb{V}(X) - \mathbb{C}(XY))\mu_Y^2 + \\
&\quad (\mathbb{V}(Y) - \mathbb{C}(XY))\mu_X^2 - \mathbb{C}(XY)^2 - \\
&\quad \mathbb{C}(XY)(\mu_X \mu_Y) - \mu_X^2 \mu_Y^2) + \mathbb{V}(e) && (64)\\
&= b_1^2(\mathbb{V}(X)\mathbb{C}(XY) - \mathbb{C}(XY)^2 + \\
&\quad \mathbb{V}(X)\mathbb{V}(Y) - \mathbb{C}(XY)\mathbb{V}(Y) - \mathbb{V}(X)\mathbb{C}(XY) + \mathbb{C}(XY)^2 + \\
&\quad \mathbb{C}(XY)\mathbb{V}(Y) - \mathbb{C}(XY)^2 + 3\mathbb{C}(XY)^2 + \\
&\quad \mathbb{V}(X)\mu_Y^2 - \mathbb{C}(XY)\mu_Y^2 + \mathbb{V}(Y)\mu_X^2 - \mathbb{C}(XY)\mu_X^2 - \\
&\quad \mathbb{C}(XY)^2 - \mathbb{C}(XY)\mu_X \mu_Y - \mu_X^2 \mu_Y^2) + \mathbb{V}(e) && (65)\\
&= b_1^2(\mathbb{V}(X)\mu_Y^2 + \mathbb{V}(Y)\mu_X^2 + \mathbb{V}(X)\mathbb{V}(Y) + \mathbb{C}(XY)^2 - \\
&\quad \mathbb{C}(XY)(\mu_Y + \mu_X)^2 - \mu_X^2 \mu_Y^2) + \mathbb{V}(e) && (66)
\end{aligned}
$$

**Expected covariance matrix and mean vector of $Z = b_1 XY + e$**

We now have all of the elements in $\mathbb{E}(\Sigma)$, the expected covariance matrix and $M$, the mean vector of the model $Z = b_1 XY + e$

$$
\mathbb{E}(\Sigma) = \begin{bmatrix}
\mathbb{V}(X) & \mathbb{C}(XY) & \begin{matrix} b_1(\mathbb{V}(X)\mu_Y + \\ \mathbb{C}(XY)\mu_X) \end{matrix} \\
\mathbb{C}(XY) & \mathbb{V}(Y) & \begin{matrix} b_1(\mathbb{V}(Y)\mu_X + \\ \mathbb{C}(XY)\mu_Y) \end{matrix} \\
\begin{matrix} b_1(\mathbb{V}(X)\mu_Y + \\ \mathbb{C}(XY)\mu_X) \end{matrix} & \begin{matrix} b_1(\mathbb{V}(Y)\mu_X + \\ \mathbb{C}(XY))\mu_Y \end{matrix} & \begin{matrix} b_1^2(\mathbb{V}(X)\mu_Y^2 + \mathbb{V}(Y)\mu_X^2 \\ +\mathbb{V}(X)\mathbb{V}(Y) + \mathbb{C}(XY)^2 \\ \mathbb{C}(XY)(\mu_Y + \mu_X)^2 \\ -\mu_X^2 \mu_Y^2) + \mathbb{V}(e) \end{matrix}
\end{bmatrix} \quad (67)
$$

$$
\mathbb{E}(M) = \begin{bmatrix} \mu_X & \mu_Y & b_1(\mathbb{C}(XY) + \mu_X \mu_Y) \end{bmatrix} \quad (68)
$$

Since $b_1$ appears in multiple cells of the expected covariance matrix, and does not only appear as a squared term this gives us the constraints necessary to determine both $b_1$ and $V_e$ using an optimizer such as maximum likelihood.

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